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The noncommutative space of stochastic diffusion systems

B Aneva

Physics Department, LMU University, D-80333 Muenchen, INRNE, Bulgarian Academy of Sciences, 1784 Sofia, Bulgaria

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Abstract
In the matrix product ground states approach to n-species diffusion processes the stationary probability distribution is expressed as a matrix product state with respect to a quadratic algebra determined by the dynamics of the process. We show that the quadratic algebra defines a noncommutative space with a $GL_q(n)$ quantum group action as its symmetry. Boundary processes account for the appearance of parameter-dependent linear terms in the algebraic relations. We argue that for systems with boundary conditions the diffusion algebras are also obtained either by a shift of basis in the n-dimensional quantum plane or by an appropriate change of basis in a lower dimensional one which leads to a reduction of the $GL_q(n)$ symmetry.

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1. Introduction

Stochastic reaction-diffusion processes are of both theoretical and experimental interest not only because they describe various mechanisms in physics and chemistry [1] but they also provide a way of modelling phenomena such as traffic flow [2], the kinetics of biopolymerization [3] and interface growth [4].

The conventional description of a stochastic process is in terms of a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable s_i taking n integer values, $s_i = 0, 1, 2, \ldots, n-1$ at each site $i = 1, 2, \ldots, L$ of a linear chain. A configuration s on the lattice at a time t is determined by the set of occupation numbers $s_1, s_2, \ldots, s_L$ and a transition to another configuration s' during an infinitesimal time step $\mathrm{d}t$ is given by the probability $\Gamma(s, s')\,\mathrm{d}t$. The time evolution of the stochastic system is governed by the master equation

$$\frac{\mathrm{d}P(s,t)}{\mathrm{d}t} = \sum_{s'} \Gamma(s,s')P(s',t) \tag{1}$$

for the probability $P(s, t)$ of finding the configuration s at a time t. These are Markov processes [5] and the rate matrices are intensity matrices with the property that the sum of elements in each column is zero (probability conservation):

$$\Gamma(s, s) = -\sum_{s' \neq s} \Gamma(s', s). \tag{2}$$

With the restriction of dynamics to changes of configuration only at two adjacent sites during the time $\mathrm{d}t$ the transition rates for such changes depend only on these sites. The two-site rates $\Gamma \equiv \Gamma^{ik}_{jl}$, where $i, j, k, l = 0, 1, 2, \ldots, n-1$, are assumed to be independent of the position in the bulk. At the boundaries, i.e. sites 1 and L, additional processes can take place with rates L^i_k and R^i_k $(i, k = 0, 1, 2, \ldots, n-1)$.

Since the master equation (1) for the time evolution of a stochastic system is linear it can be mapped to a Schrödinger equation for a quantum Hamiltonian in imaginary time. In this mapping the probability distribution becomes a ket state $|P(t)\rangle$ in a vector space with an orthonormal basis and the master equation reads

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = -HP(t) \tag{3}$$

where

$$H = \sum_j H_{j,j+1} + H^{(L)} + H^{(R)} \tag{4}$$

H is defined in terms of the transition rates and describes nearest-neighbour interaction in the bulk with single-site boundary terms. The ground state of this in general non-Hermitean Hamiltonian corresponds to the stationary probability distribution of the stochastic dynamics. It has zero energy by construction. (H is an intensity matrix with positive rates.) For many reaction-diffusion stochastic systems this mapping provides a connection with integrable quantum spin chains [6] and allows one to treat the stochastic dynamics with the formalism of quantum mechanics. Well known examples of diffusion-type systems are the two-species symmetric exclusion process [7] and the diffusion-driven one [8] both with a hard core repulsion which prevents the occupation of a lattice site by more than one particle. In the symmetric case particles hop between lattice sites i, j with rates $g_{ij} = g_{ji}$. The stochastic Hamiltonian is identical to the $SU(2)$ symmetric spin-$\frac{1}{2}$ isotropic Heisenberg ferromagnet [9]

$$H = -\tfrac{1}{2} \sum_i (\sigma^x_i \sigma^x_j + \sigma^y_i \sigma^y_j + \sigma^z_i \sigma^z_j - 1). \tag{5}$$

The $SU(2)$ symmetry, yet unrevealed in the original master equation becomes manifest through the mapping and allows for exact results of the stochastic dynamics. The case of a diffusion-driven lattice gas of particles in one dimension with rates $\frac{g_{i,i+1}}{g_{i+1,i}} = q \neq 1$ is mapped to a $SU_q(2)$-symmetric XXZ chain with anisotropy $\Delta = \frac{(q+q^{-1})}{2}$.

A different approach to stochastic dynamics is the matrix product states description [10, 11] of the ground state wavefunction of the quantum Hamiltonian, i.e. the stationary probability distribution, as a product of (or a trace over) matrices. These matrices form a representation of an algebra determined by the dynamics of the process. The approach was introduced by Derrida *et al* [10] who formulated an exactly solved model of non-equilibrium physics in terms of an infinite-dimensional representation of the stationary, matrix algebra for the two-species asymmetric exclusion process with open boundaries. It was further applied to three-species diffusion systems on a ring [12] and with open boundaries [13] and to reaction-diffusion processes as well [14].

The advantage of the matrix product state method is that important physical properties and quantities such as multiparticle correlation functions, currents, density profiles and phase diagrams can be obtained from the representations of the matrix quadratic algebra. Despite the extended study of simple generalizations of the exclusion model exact solutions for systems with several species of particles are still lacking. In a recent paper [15] quadratic algebras with a Poincare–Birkhoff–Witt (PBW) basis corresponding to n-species diffusion systems have been classified. The known examples of physical applications are obtained as special cases in this classification. The aim of this work is to study the representations of the n-species quadratic diffusion algebras.

We consider a diffusion process with n species on a chain with L sites with nearest-neighbour interaction with exclusion, which means that a site can be either occupied by one particle or empty. The empty site is referred to as a vacancy (or a hole), the rest $n-1$ species—as different types of particles. In the set of occupation numbers $(s_1, s_2, \ldots, s_L)$ specifying a configuration of the system $s_i = 0$ if there is a vacancy at a site i, $s_i = 1$ if there is a first-type particle at a site $i, \ldots, s_i = n-1$ if there is an $(n-1)$th-type particle at a site i. Diffusion or Brownian motion is the oldest physical example of a Markov process corresponding to a probability rate matrix $\Gamma^{ik}_{ki} = g_{ik}$, with $i, k = 0, 1, 2, \ldots, n-1$. On successive sites the species i and k exchange places during every infinitesimal time interval $\mathrm{d}t$ with probability $g_{ik}\,\mathrm{d}t$. The event of exchange happens if out of two adjacent sites one is a vacancy and the other is occupied by a particle of a given type, or each of the sites is occupied by one particle of a different type; otherwise the exchange is rejected. The simplest form is the n-species symmetric exclusion process known as the lattice gas model when each particle hops between nearest-neighbour sites with a constant rate. Closely related is the n-species asymmetric exclusion process of particles hopping in a preferred direction. An example of such a system is the diffusion-driven lattice gas where particles move under the action of an external field. The driving force imposes a bias on the hopping rates so that the process is totally asymmetric if all jumps are in one direction only (forward) and partially asymmetric if there is a non-zero probability of also hopping backward, the rates g_{ik} for moving to the left being different from the rates g_{ki} for moving to the right. The number of particles n_i of each species in the bulk is conserved:

$$\sum_{i=0}^{n-1} n_i = L. \tag{6}$$

One distinguishes closed systems (periodic boundary conditions) and open systems with boundary processes. In the case of periodic boundary conditions one treats the site index as cyclic, $s_{L+i} = s_i$ to obtain a system on a ring. The number of particles of each type remains conserved. Boundary processes break the $n-1$ conservation laws in the bulk. The general form of the boundary processes is that at site 1 (the left-hand side of the chain) and at site L (the right-hand side of the chain) the particle i is replaced by the particle k with probabilities $L^i_k\,\mathrm{d}t$ and $R^i_k\,\mathrm{d}t$, respectively. The matrices L and R are such that the sum of the diagonal elements in each column with the positive non-diagonal ones is zero:

$$L^i_i = -\sum_{j=0}^{n-1} L^i_j \qquad R^i_i = -\sum_{j=0}^{n-1} R^i_j. \tag{7}$$

In most studied examples of the two-species symmetric or asymmetric exclusion process the lattice gas is coupled to external reservoirs of particles of fixed density to account for the additional degrees of freedom at the boundaries. A particle is injected into site 1 and removed from site L with corresponding constant rates. The choice of such boundary processes is motivated by the fact that they can induce phase transitions [16]. A generalization to the boundary processes of an open diffusion system with n species is that the particle of type k is introduced with a rate L^0_k and/or removed with a rate L^k_0 at the left end of the chain and it is removed with a rate R^k_0 and/or introduced with a rate R^0_k at the right, with $k = 1, 2, 3, \ldots, n-1$.

Through different physical interpretation the diffusion processes cover a wide range of phenomena. The symmetric simple exclusion process is the lattice gas model for reptation [17] of single polymer chains in a random environment of other polymers. The diffusive motion of polymer segments (defects) is similar to lattice Brownian motion (particle–hole exchange) with an exclusive interaction and even though one dimensional it is used to describe a three-dimensional system. Physical application of the two-species asymmetric exclusion process are systems where it is important to understand the current of particles through channels of finite length such as diffusion-driven lattice gas models of biopolymerization kinetics, fluctuations of interfaces and traffic flow. The multi-species asymmetric exclusion process provides a simplified model of traffic flow and by joining many open-boundary systems into a network appears to describe realistic traffic phenomena [18]. The intensive study of the asymmetric exclusion process is further motivated due to its connection with interface dynamics since it can be exactly mapped to a 1 + 1 model of interface growth [19] where a particle at a site corresponds to a step downwards of one unit of growth and a hole at a site corresponds to a step upwards.

In the matrix product states approach [20] to a n-species process one considers n matrices D_i acting in an auxiliary vector space and satisfying the algebra

$$g_{ik} D_i D_k - g_{ki} D_k D_i = x_k D_i - x_i D_k \tag{8}$$

where g_{ik} and g_{ki} are positive (or zero) probability rates, x_i are c-numbers and $i, k = 0, 1, \ldots, n-1$. (No summation over repeated indices in equation (8).) For systems with periodic boundary conditions the probability distribution is given by the expression

$$P(s_1, \ldots, s_L) = \mathrm{Tr}\,(D_{s_1} D_{s_2} \ldots D_{s_L}). \tag{9}$$

When boundary processes are considered the probability distribution is given by a matrix element in the auxiliary vector space

$$P(s_1, \ldots, s_L) = \langle w | D_{s_1} D_{s_2} \ldots D_{s_L} | v \rangle \tag{10}$$

and the vectors $|v\rangle$ and $\langle w|$ are determined by the conditions

$$\langle w | (L_i^k D_k + x_i) = 0 \qquad (R_i^k D_k - x_i) | v \rangle = 0 \tag{11}$$

where due to (7)

$$\sum_{i=0}^{n-1} x_i = 0. \tag{12}$$

Thus to find the stationary probability distribution one has to compute traces or matrix elements with respect to the vectors $|v\rangle$ and $\langle w|$ of monomials of the form

$$D_{s_1}^{m_1} D_{s_2}^{m_2} \ldots D_{s_L}^{m_L}. \tag{13}$$

The problem to be solved is twofold. For periodic boundary conditions one looks for representations of the matrix algebra (8) consistent with the recurrence relations that are enforced by the algebraic relations. In the case of open systems one has to find matrix representations consistent with the boundary conditions (11), namely that the combinations $(L_i^k D_k + x_i)$ and $(R_i^k D_k - x_i)$ in (11) have common eigenvectors with eigenvalue zero.

The relations (8) allow an ordering of the elements D_k. Monomials of given order are the PBW basis for polynomials of fixed degree as the probability distribution is due to the conservation laws (6). We consider an associative algebra generated by a unit e and n elements D_k obeying $n(n-1)/2$ relations (8). The alphabetically ordered monomials

$$D_{s_1}^{n_1} D_{s_2}^{n_2} \ldots D_{s_l}^{n_l} \tag{14}$$

where $s_1 < s_2 < \cdots s_l$, $l \geqslant 1$ and $n_1, n_2, \ldots, n_l$ are non-negative integers, are a linear basis in the algebra. The possibility of alphabetical ordering is achieved using the relations (8). It is sufficient to verify coincidence of two different ways of ordering for cubic monomials only which gives a relation for the rates g_{ik}. The ordering of higher order monomials does not give rise to any further relation. The linear independence of alphabetically ordered monomials is also sufficient to verify for cubic monomials only [21].

The algebra (8) admits an involution through the mapping $D_i \rightarrow D_i^+$ which with real parameters $x_i = \bar{x_i}$ defines Hermitean elements $D_i = D_i^+$ provided

$$g_{ij}^+ = -g_{ji} \tag{15}$$

(or anti-Hermitean $D_i = -D_i^+$ if $g_{ij}^+ = g_{ji}$).

A PBW property of an algebra is important when one considers a quantum group of transformations of a quantum plane and a deformation of a universal enveloping as its dual. The quantum transformations and duality properties are written in a compact form with an R-matrix operator satisfying a quantum Yang–Baxter equation. Triangular Yang–Baxter relations are coded expressions for hidden symmetries of integrable models.

It is readily seen that the relations (8) with all x_k equal to zero are the defining relations of a $GL_q(n)$ quantum plane to which a multiparameter R matrix corresponds. The presence of the linear terms on the right-hand side of (8) with different from zero c-numbers x_k will break the quantum invariance but not completely. Due to the requirement that the braid associativity condition is also fulfilled with the linear terms the symmetry will be only reduced. On the other hand when all the rates g_{ik} are equal the relations (8) are of a Lie-algebra type. The n generators D_i can be mapped to the generators of $GL(n)$.

The origin of the hidden symmetries in the matrix algebra might have important consequences as a step to integrability of the diffusion stochastic dynamics. A model with $GL_q(n)$ symmetry has an R-matrix operator, a constant solution of the Yang–Baxter equation. Through a 'Baxterization' procedure, originally introduced by Jones [22], one can construct the matrix $\check{R}(\lambda)$, a parameter-dependent solution of the Yang–Baxter equation. The spectral parameter λ is a suitably chosen function of the temperature, field strengths or other physical parameters of the model. For integrable quantum spin chain models one can build a one-parameter family of commuting transfer matrices $[T(\lambda), T(\mu)] = 0$, a property directly implied by the Yang–Baxter equation. The importance of this commutativity becomes clear from the expansion of the transfer matrix $\ln T(\lambda) = \sum_n \frac{\lambda^n}{n!} Q_n$ which yields an infinite set of mutually commuting conserved charges Q_n including the Hamiltonian. As mentioned above much progress has been achieved in understanding the dynamics of the two-species symmetric and asymmetric simple exclusion process through the mapping to the XXX and XXZ integrable spin chains with $SU(2)$ and $SU_q(2)$ symmetry, respectively. In both cases there exists a conserved charge which is one of the generators of the corresponding symmetry and expresses particle number conservation.

In this paper we obtain representations of the algebra (8) that are consistent with the boundary conditions and reveal the hidden symmetry algebras of the corresponding diffusion processes. The quadratic algebra associated with the n-species symmetric and asymmetric diffusion processes defines in general the comodule structure of $GL(n)$ and $GL_q(n)$. The hidden symmetries imply a possibility for a mapping to the $GL(n)$ integrable quantum spin chain associated with an n-state vertex model [23], and in particular the six-vertex model for $n = 2$. The number of species n are the spin states n of a spin variable S related by $n = 2S + 1$. The most important constraint imposed by the symmetry is the charge (spin momentum) conservation interpreted as particle number conservation. The introduction of boundary processes reduces the $GL_q(n)$ symmetry in the bulk to either $GL_q(n-1)$ or $GL_q(n-2)$, thus breaking the charge (particle number) conservation law.

Proposition.

(1) In the case of a Lie-algebra type diffusion algebras the n generators D_i, and e can be mapped to the generators J_{jk} of $GL(n)$ and the mapping is invertible. The universal enveloping algebra generated by D_i belongs to the universal enveloping algebra of the Lie algebra of $GL(n)$.

(2) It is known that the multiparameter quantized noncommutative space can be realized equivalently as a q-deformed Heisenberg algebra [24] of n oscillators depending on $n(n-1)/2+1$ parameters (or in general on $n(n-1)/2+n$ parameters [25]). The universal enveloping algebra (UEA) of the elements D_i in the case of a diffusion algebra with all coefficients x_i on the right-hand side of equation (8) equal to zero belongs to the UEA of a multiparameter deformed Heisenberg algebra to which a consistent multiparameter $GL_q(n)$ quantization corresponds.

(3) It has been shown in [15] that for a relation with non-zero x-terms on the right-hand side of (8) only then is braid associativity satisfied if out of the coefficients x_i, x_k, x_l corresponding to an ordered triple $D_i D_k D_l$ either one coefficient x is zero or two coefficients x are zero and the rates are respectively related. We present a solution for the corresponding algebraic relations in terms of deformed oscillators. We argue that the appearance of the non-zero linear terms in the right-hand side of the quantum plane relations leads to a lower dimensional noncommutative space and a reduction of the $GL_q(n)$ invariance. We show that the diffusion algebras in this case can be obtained by either a change of basis in the n-dimensional noncommutative space or by a suitable change of a subsystem of the basis of the lower dimensional quantum space.

We construct infinite- and/or finite-dimensional representations for the cases listed above. The representations are consistent with the corresponding boundary conditions. We give the solutions of the algebraic relations for general n and comment on the invariance properties. The boundary vectors are obtained explicitly for $n = 2$ and 3 and a generalization to higher n if not straightforward is discussed in each case.

We first note that the algebra (8) always has the one-dimensional representations with the corresponding relations for the rates. These have been much commented upon in the literature and will not be the subject of this paper. We point out again that the probability rates g_{ij} and g_{ji} are both positive or only $g_{ij} = 0, i < j$ which defines different algebras corresponding to particular processes. Even though not explicitly emphasized the positivity of the parameters g_{ij} implies that the symmetries in point are the real forms of the classical and quantum $GL(n)$ which are induced by the reality structures on the representation space according to the involution in the algebra. We proceed now with the construction of the representations of the algebra other than the one-dimensional ones.

2. Lie-algebra types

There are two such algebras. The first type is when all rates are equal and corresponds to the n-species symmetric exclusion process. The second algebra (with g_{ki} only) appears in the description of the multispecies totally asymmetric exclusion process.

2.1. All rates equal, $g_{ij} = g_{ji} = g$

The algebra after rescaling the generators D_i, $i = 0, 1, 2, \ldots, n-1$, by

$$D_i = \frac{x_i}{g} D_i' \qquad \sum_{i=1}^{n-1} x_i = 0 \tag{16}$$

takes the form (the primes are omitted)

$$\begin{aligned}
&[D_0, D_1] = D_0 - D_1\\
&[D_0, D_2] = D_0 - D_2\\
&\vdots\\
&[D_{n-2}, D_{n-1}] = D_{n-2} - D_{n-1}.
\end{aligned} \tag{17}$$

These algebraic relations are solved in terms of the $GL(n)$ Lie-algebra generators J_i^j:

$$\begin{aligned}
&D_0 = J_0^0 + J_0^1 + J_0^2 + \cdots + J_0^{n-1}\\
&D_1 = J_1^0 + J_1^1 + J_1^2 + \cdots + J_1^{n-1}\\
&D_2 = J_2^0 + J_2^1 + J_2^2 + \cdots + J_2^{n-1}\\
&\vdots\\
&D_{n-1} = J_{n-1}^0 + J_{n-1}^1 + J_{n-1}^2 + \cdots + J_{n-1}^{n-1}.
\end{aligned} \tag{18}$$

The conventional basis for the fundamental representation of the $GL(n)$ generators given by the $(e_{ij})_{ab} = \delta_{ia}\delta_{jb}$, $i, j, a, b = 0, 1, 2, \ldots, n-1$, provides the n-dimensional matrix representation of the generators D_i, with entries 1 in only the first row of D_0, the second row of D_1, the third row of $D_3, \ldots$ the last row of D_{n-1} and all the entries elsewhere zero. The correspondence is one-to-one since the $GL(n)$ Lie-algebra generators can be expressed with the help of the transposed matrices, namely

$$J_i^j = \frac{1}{n} D_i D_j^T \tag{19}$$

where $i, j = 0, 1, 2, \ldots, n-1$. The PBW basis of the algebra generated by the elements D_i thus belongs to the basis of the universal enveloping algebra of $sl(n) \oplus u(1)$ and this is the hidden symmetry algebra of a stochastic diffusion system with all rates equal.

We have to further show that the algebraic solution is compatible with the boundary conditions. The boundary eigenvalue problem turns to be different for $n = 2$ and $n > 2$ and therefore we discuss in detail both the $n = 2$ and 3 cases.

2.1.1. The algebra and boundary problem for $n = 2$ and 3. The algebra generated by the two matrices D_0 and D_1, obeying $[D_0, D_1] = D_0 - D_1$, is solved by substituting D_0 and D_1 from (18) expressed, respectively, in terms of the $GL(2)$ generators. The boundary vectors are determined by the conditions

$$\langle w|(L_1^0 D_0 - L_0^1 D_1 + x_1) = 0 \tag{20}$$

$$(-R_1^0 D_0 + R_0^1 D_1 - x_0)|v\rangle = 0 \tag{21}$$

with $x_0 + x_1 = 0$. With the expressions for the matrices D_0 and D_1 in terms of the $GL(2)$ generators the boundary matrices are simultaneously diagonalized with the constraints

$$L_1^0 + L_0^1 = g \qquad R_0^1 + R_1^0 = -g \tag{22}$$

which is a contradiction since all the rates are probability rates and therefore have to be positive. There is, however, an algebraic solution for the matrices D that is consistent with the boundary conditions, namely

$$\begin{aligned} D_0 &= \frac{x_0}{g}((1+\alpha)J_0^0 + J_0^1 + \alpha J_1^1) \\ D_1 &= \frac{x_1}{g}(\alpha J_0^0 + J_1^0 + (1+\alpha)J_1^1). \end{aligned} \tag{23}$$

It introduces an additional arbitrary parameter and this is the price to be paid to match the algebra with the boundary vectors which hence determines a Fock representation of the diffusion algebra with a constraint for the rates

$$g(L_0^1 + L_1^0 + R_0^1 + R_1^0) = (L_0^1 + L_1^0)(R_0^1 + R_1^0). \tag{24}$$

Unlike the $n = 2$ problem the expressions for the $n = 3$ D-matrices

$$\begin{aligned} D_0 &= \frac{x_0}{g}(J_0^0 + J_0^1 + J_0^2) \\ D_1 &= \frac{x_1}{g}(J_1^0 + J_1^1 + J_1^2) \\ D_2 &= \frac{x_2}{g}(J_2^0 + J_2^1 + J_2^2) \end{aligned} \tag{25}$$

that solve the diffusion algebra yield a consistent solution for the boundary vectors. The latter are in this case determined by the systems

$$\begin{aligned} &\langle w|((-L_1^0 - L_2^0)D_0 + L_0^1 D_1 + L_0^2 D_2 + x_0) = 0 \\ &\langle w|(L_1^0 D_0 + (-L_0^1 - L_2^1)D_1 + L_1^2 D_2 + x_1) = 0 \\ &\langle w|(L_2^0 D_0 + L_2^1 D_1 + (-L_0^2 - L_1^2)D_2 + x_2) = 0 \end{aligned} \tag{26}$$

and

$$\begin{aligned} &((-R_1^0 - R_2^0)D_0 + R_0^1 D_1 + R_0^2 D_2 - x_0)|v\rangle = 0 \\ &(R_1^0 D_0 + (-R_0^1 - R_2^1)D_1 + R_1^2 D_2 - x_1)|v\rangle = 0 \\ &(R_2^0 D_0 + R_2^1 D_1 + (-R_0^2 - R_1^2)D_2 - x_2)|v\rangle = 0 \end{aligned} \tag{27}$$

with

$$x_0 + x_1 + x_2 = 0. \tag{28}$$

The parameters x provide a matching condition for a common eigenvalue zero of the linear forms of the left and right transition matrices with the corresponding left and right boundary vectors and constraints on the boundary rates

$$\begin{aligned} &R_0^1 L_0^2 + L_0^1 R_0^2 + (L_1^0 + L_2^0)(R_0^1 + R_0^2) + (R_1^0 + R_2^0)(L_0^1 + L_0^2) \\ &\qquad = g(L_0^1 - L_0^2 + R_0^1 - R_0^2) \\ &(R_0^1 + R_2^1)L_1^2 - (L_0^1 + L_2^1)R_1^2 + R_1^0(L_0^1 + L_2^1 + L_1^2) - L_1^0(R_1^2 + R_0^1 + R_2^1) \\ &\qquad = g(L_1^0 - L_1^2 + R_1^0 - L_1^2). \end{aligned} \tag{29}$$

The generalization of these representations to general n is straightforward. One can also use a realization of the $GL(n)$ operators in terms of creation and annihilation operators

$$J_{ik} = A_i^+ A_k \tag{30}$$

to obtain a representation of the elements D and the the boundary vectors in the oscillator basis.

2.2. *Algebras with only $g_{ji} \neq 0$*

The algebraic relations read

$$g_{ji} D_j D_i = x_i D_j - x_j D_i \tag{31}$$

with $i, j = 0, 1, 2, \ldots, n-1$ and $j > i$. This algebra has been considered in [26]. We present here a different solution. To find the general solution of this algebra we consider the matrix

$$D = \begin{pmatrix} 1 & -a_1 & 0 & \ldots & 0 \\ 0 & 1 & -a_2 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & 0 & \ldots & 1 \end{pmatrix} \tag{32}$$

and its transposed

$$D^T = \begin{pmatrix} 1 & 0 & 0 & \ldots & -a_n \\ -a_1 & 1 & 0 & \ldots & 0 \\ 0 & -a_2 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \tag{33}$$

where $\prod_{i=1}^{n} a_i = 1$. These matrices satisfy

$$DD^T = D + D^T \tag{34}$$

with $a_i^2 = 1$. This equation is also valid if all the parameters a_i are equal, namely

$$D(a_i)D^T(a_j) = D(a_i) + D^T(a_j) \tag{35}$$

and $a_i a_j = 1$. We rescale the elements D_i for $i = 1, 2, \ldots, n-1$ and D_0 by

$$D_i = x_i D'_i \qquad D_0 = x_0 D'_0 \tag{36}$$

and rewrite the algebraic relations in the form (the primes are omitted)

$$g_{i0} D_i D_0 = D_i - D_0 \tag{37}$$

$$g_{ji} D_j D_i = D_j - D_i \tag{38}$$

where $i, j, k = 1, 2, \ldots, n-1$. We then associate with D_0 and D_i a given value of the parameter a of the matrices D in (32) and use (37) to express the matrices D_i and D_i^T in terms of D_0

$$\begin{aligned} D_i(a_i) &= D_0(a_i)(1 - g_{i0} D_0(a_i))^{-1} \\ D_i^T(a_i) &= (1 - g_{i0} D_0^T(a_i))^{-1} D_0^T(a_i) \end{aligned} \tag{39}$$

the matrices $(1 - g_{i0} D_0(a_i))$ and $(1 - g_{i0} D_0^T(a_i))$ being invertible. Equation (31) is solved with the substitution

$$D_i \equiv D_i(a_i) \qquad D_j \equiv D_j^T(a_j) \qquad i, j = 1, 2, \ldots, n-1 \tag{40}$$

and the assumption

$$D_0 = D_0^T \tag{41}$$

which is verified by inserting expression (39) in (38). One finds

$$(g_{ji} - g_{j0} + g_{i0}) D_0^T(a_j) D_0(a_i) = D_0^T(a_j) - D_0(a_i). \tag{42}$$

Due to

$$D_0(a_i) = D_0^T(a_j) \tag{43}$$

one has

$$(g_{ji} - g_{i0} + g_{j0}) D_0^T(a_j) D_0(a_i) = 0 \tag{44}$$

which gives a constraint for the rates

$$g_{ji} = g_{j0} - g_{i0} \tag{45}$$

a particular case of which is the one-dimensional representation. A realization of a Hermitean D_0 is given by means of the matrices D and D^T in (32) and (33):

$$D_0 = D + D^T. \tag{46}$$

The symmetry compared to the case when all g_{ij} and g_{ji} are non-zero is reduced from $SL(n) \otimes U(1)$ to $U(1)^{n-1} \otimes U(1)$.

We note that the solution we have considered is in terms of commuting matrices. It is known [11] that such 'disorder' solutions correspond to the absence of correlations between the different states. However, their study is of interest since they provide insights into the phase diagram of complex systems and shed light on unexpected symmetries.

2.2.1. Boundary problem.

The case $n = 2$. The algebra

$$g_{10} D_1 D_0 = x_0 D_1 - x_1 D_0 \qquad x_0 + x_1 = 0 \tag{47}$$

is solved by

$$D_0 = \frac{x_0}{g_{10}} D \qquad D_1 = -\frac{x_1}{g_{10}} D^T \tag{48}$$

where

$$D = \begin{pmatrix} 1 & a_1 \\ a_2 & 1 \end{pmatrix} \tag{49}$$

and $a_1^2 = a_2^2 = 1$. The condition for Hermitean matrices generating $U(1)$

$$D = D^T \tag{50}$$

is equivalent to $a_1 = a_2$. The solution does not contradict the boundary conditions (20) and (21) and the boundary matrices are simultaneously diagonalized provided the boundary rates are constrained by

$$L_1^0 - L_0^1 + R_0^1 - R_1^0 = g_{10}. \tag{51}$$

The symmetry of the process is $U(1) \otimes U(1)$.

The case $n > 2$. There is no principle difference to generalize the above procedure of a consistent simultaneous diagonalization with the corresponding boundary vectors of the boundary matrices for any n. We spare the details of the long calculations and the rather cumbersome expression for the constraint of the boundary rates. We only give a slightly different solution for $n = 3$ when in contrast to the above-described general solution the matrix D_0 is expressed in terms of D_1 and D_2. The algebraic relations with three elements are

$$\begin{aligned} g_{10} D_1 D_0 &= x_0 D_1 - x_1 D_0 \\ g_{20} D_2 D_0 &= x_0 D_2 - x_2 D_0 \\ g_{21} D_2 D_1 &= x_1 D_2 - x_2 D_1 \end{aligned} \tag{52}$$

with $x_0 + x_1 + x_2 = 0$ and the relation for the rates

$$g_{21} = g_{20} - g_{10}. \tag{53}$$

A three-dimensional representation in terms of the matrix D and the transposed one D^T, where

$$D = \begin{pmatrix} 1 & -a_1 & 0 \\ 0 & 1 & -a_2 \\ -a_3 & 0 & 1 \end{pmatrix} \tag{54}$$

and

$$a_1^2 = a_2^2 = a_3^2 = 1 \qquad a_1 a_2 a_3 = 1 \tag{55}$$

is a solution of the algebraic relations (52) with

$$D_1 = \frac{x_1}{g_{21}} D^T \qquad D_2 = -\frac{x_2}{g_{21}} D \tag{56}$$

and

$$D_0 = x_0 (g_{20} D - g_{21})^{-1} D = x_0 (g_{10} D^T + g_{21})^{-1} D^T. \tag{57}$$

The last equality is fulfilled due to the constraint (45) for the rates. The symmetry algebra in this case is $u(1) \oplus u(1) \oplus (1)$.

3. The quantized noncommutative space of a diffusion model

The algebra (8) in its general form with both g_{ij} and g_{ji} different from zero is useful for the study of the partially asymmetric n-species exclusion process. One distinguishes the cases with x-dependent linear in D terms on the right-hand side of the algebraic relations (8) and with no such terms present.

3.1. Algebras with no x-dependent linear terms

The algebraic relations (8) without the x-terms on the right-hand side roughly parallel the Manin's multiparameter quantized space [21] with the n elements D_i viewed as its co-ordinates

$$g_{ij} D_i D_j - g_{ji} D_j D_i = 0 \tag{58}$$

where $i, j = 0, 1, 2, \ldots, n-1$. A representation of the quantum space is obtained by identifying the monomials (13) with the states of n oscillators

$$a_0^{+n_0} a_1^{+n_1} \ldots a_{n-1}^{+n_{n-1}} |0\rangle. \tag{59}$$

Then the generators D_i correspond to n creation operators and the noncommutative space is equivalent to a multideformed Heisenberg algebra [24]. It is convenient to consider the ratios

$$q_{ij} = \frac{g_{ij}}{g_{ji}} \qquad i < j \tag{60}$$

as the set of $n(n-1)/2$-independent parameters and introduce further n real parameters r_i. The latter are at this stage auxiliary parameters needed for a consistent quantized phase space calculus whose realization analogous to [25] proceeds as follows. One starts with n classical oscillators A_i and A_i^+, obeying $[A_i, A_j^+] = \delta_{ij}$ for $i, j = 0, 1, \ldots, n-1$ and defines $n_k = A_k^+ A_k$. A deformation of the Heisenberg algebra is achieved through the invertible maps

$$\begin{aligned} a_i &= \prod_{k>i} q_{ik}^{\frac{n_k}{2}} \sqrt{\frac{r_i^{n_i+1} - 1}{(r_i - 1)(n_i + 1)}} A_i \\ a_i^+ &= \prod_{k>i} q_{ik}^{-\frac{n_k}{2}} \sqrt{\frac{r_i^{n_i+1} - 1}{(r_i - 1) n_i}} A_i^+ \end{aligned} \tag{61}$$

where

$$\frac{r_i^{n_i}-1}{r_i-1} = a_i^+ a_i. \tag{62}$$

These deformed oscillators obey the following algebraic relations:

$$\begin{aligned} &a_i a_i^+ - r_i a_i^+ a_i = 1 \\ &a_i^+ a_j^+ - q_{ji} a_j^+ a_i^+ = 0 \\ &a_i a_j - q_{ji} a_j a_i = 0 \\ &a_i a_j^+ - q_{ji}^{-1} a_j^+ a_i = 0 \end{aligned} \tag{63}$$

with $i < j$ and $q_{ji} = q_{ij}^{-1}$. The deformed Heisenberg algebra (63) is invariant with respect to a $*$ involution

$$(a_i)^* = a_i^+ \qquad (a_i^+)^* = a_i \qquad (q_{ij})^* = q_{ij}^{-1} \qquad (r_i)^* = r_i. \tag{64}$$

A solution of the diffusion algebra (58) is obtained upon identifying

$$D_i = a_i^+ \qquad D_i^+ = a_i \qquad i = 0, 1, 2, \ldots, n-1 \tag{65}$$

which in view of (64) and (60) is consistent with the algebra conjugation property (15). The noncommutative space spanned by the matrices D_i is hence extended to the equivalent form of a multideformed Heisenberg algebra (63).

The deformed oscillators can be arranged in bilinears to construct the infinitesimal generators of the quantum $GL_q(n)$-group action

$$J_{ij} = a_i^+ a_j \tag{66}$$

or equivalently

$$J_{ij} = D_i D_j^+ \tag{67}$$

$i, j = 0, 1, \ldots, n-1$. A consistent multiparameter $GL_q(n)$ quantization [27, 28] depending on $n(n-1)/2$-independent parameters is achieved with all auxiliary r_i set equal to one (which makes the quantum phase space (63) coincide with the $GL_q(n)$ covariant formulation of the quantum plane by Wess and Zumino [29]). The corresponding R-matrix satisfies an Yang–Baxter equation and is diagonal. If we want to relate such an R-matrix to a transition probability rate matrix we are lead to a trivial matrix. Therefore one should consider the alternative case of a covariant $GL_q(n)$ quantization with all deformation parameters q_{ij} and r_i equal, i.e.

$$q_{ij} = q = r_i^{-1} \qquad i < j \qquad i, j = 0, 1, 2, \ldots, n-1 \tag{68}$$

which amounts to a one-parameter deformation of the UEA of $gl(n)$, with $q > 0$. The UEA generated by the n elements D_i belongs to the quantum algebra $U_q(gl(n))$.

A representations of the matrices D_i and D_i^+ and the generators $J_{ij}, i, j = 0, 1, 2 \ldots n-1$, is provided by the action of the oscillators a_i^+ and a_i on the basis vectors $|n_0 n_1 \ldots n_{n-1}\rangle$ (with $n_i = 0, 1, 2 \ldots$)

$$a_i^+ |n_0 n_1 \ldots n_{n-1}\rangle = \frac{(1-q^{n_i+1})^{1/2}}{(1-q)^{1/2}} |n_0 n_1 \ldots n_{i-1}, n_i + 1, n_{i+1} \ldots n_{n-1}\rangle \tag{69}$$

$$a_i |0\rangle = 0 \qquad i = 0, 1, \ldots, n-1 \tag{70}$$

$$a_i |n_0, n_1 \ldots n_{n-1}\rangle = \frac{(1-q^{n_i})^{1/2}}{(1-q)^{1/2}} |n_0 n_1 \ldots n_{n-1}, n_i - 1, n_{i+1} \ldots n_{n-1}\rangle. \tag{71}$$

Thus a diffusion algebra with no x-dependent terms is a quadratic algebra, equivalent to a noncommutative space with a $GL_q(n)$ quantum group action as its symmetry and with the associated R-matrix operator satisfying the Yang–Baxter equation. The infinitesimal operators of the quantum group action generate the quantum algebra $U_q(gl(n))$ (denoted $gl_q(n)$ hereafter) which is the hidden symmetry algebra of the model.

3.2. Algebras with x-dependent linear terms

As mentioned above when x-dependent terms are present on the right-hand side of equation (8) braid associativity is satisfied iff out of the coefficients x_i, x_k, x_l related to the ordered triple $D_i D_k D_l$ either one x or two x are zero and the rates are correspondingly constrained. We proceed now with our argument to show that in both cases the algebraic relations can be reduced from the quantum plane relations.

3.2.1. Algebras with one x-dependent linear term. The relation to a quantum space of a diffusion algebra with a linear term on the right-hand side depending on one parameter x only has been also noted in [15]. We discuss the correspondence from the point of view of invariance properties.

We consider the quantum plane

$$D_i D_j = q_{ji} D_j D_i \qquad i < j \qquad q_{ij} q_{ji} = 1 \tag{72}$$

where $i, j = 0, 1, 2, \ldots, n-1$ and shift one of the generators D_i, fixing the index of the rescaled element to be $i \equiv i_1$ (it can be any one of the matrices D)

$$D_{i_1} = D'_{i_1} + x_{i_1}/(1 - g_{i_1}) \tag{73}$$

with $q_{i_1} = q_{ji_1}$ for j running all the remaining $n-1$ indices. Due to the shift the quommutators containing the rescaled element are only changed

$$D_{i_1} D_j = q_{ji_1} D_j D_{i_1} \tag{74}$$

and we obtain the algebra of the diffusion ground state matrices in the form

$$D_{i_1} D_j - q_{i_1} D_j D_{i_1} = -x_{i_1} D_j \qquad i_1 \neq j \tag{75}$$

$$D_j D_k = q_{kj} D_k D_j \tag{76}$$

where j, k run the set of all the indices of the $n-1$ elements D obtained upon exclusion of the index of the shifted one. One can also write the algebra in the form

$$\begin{aligned} &g_{i_1 j} D_{i_1} D_j - g_{ji_1} D_j D_{i_1} = -x_{i_1} D_j \\ &g_{jk} D_j D_k - g_{kj} D_k D_j = 0 \qquad i_1 \neq j, k. \end{aligned} \tag{77}$$

Since the quantum plane is invariant with respect to co-ordinate translations if we keep all the deformation parameters equal, i.e. $q_{ji_1} = q_{jk} = q_{i_1}$, we still have the quantum group $GL_q(n)$ as the symmetry of the shifted plane. The symmetry is reduced if $q_{ji_1} \neq q_{jk}$. The algebraic relations (76) define a noncommutative space of dimension $n-1$. A representation is constructed in terms of $n-1$ deformed creation operators a_j^+ according to the procedure outlined in the previous section upon identifying $D_j = a_j^+$ for j taking $n-1$ values and $j \neq i_1$. With the constraint

$$\frac{g_{jk}}{g_{kj}} = q > 0 \qquad j < k \tag{78}$$

the oscillator algebra leads to a consistent one-parameter $GL_q(n-1)$ quantization and a deformation of the UEA of $gl(n-1)$ with a corresponding R-matrix satisfying the Yang–Baxter equation. The rescaled matrix D_{i_1} forms a $u(1)$ algebra. Thus the presence of one x-dependent term due to a boundary process reduces the $GL_q(n)$ invariance in the bulk to $GL_q(n-1) \otimes U(1)$ invariance.

3.2.2. Algebras with two x-dependent linear terms. A diffusion algebra with two c-numbers x always contains one (and only one) relation of the form

$$g_{ij}D_iD_j - g_{ji}D_jD_i = x_jD_i - x_iD_j. \tag{79}$$

This can be mapped to one of the quommutators of the Heisenberg algebra, namely

$$a_ia_i^+ - r_ia_i^+a_i = 1 \tag{80}$$

by a simultaneous shift of the creation and the annihillation operators. Thus to make the correspondence with the quantum plane one has to keep in mind that one of the generators of a diffusion algebra with two x-terms has to be identified with a rescaled annihilation operator. Therefore one considers a quantum plane of dimension $n-1$ and the reduction to a diffusion algebra proceeds as follows. Since in equation (79) $i < j$ it is convenient to choose $i = 0$ and $j = n-1$ and rewrite the relation in the form

$$D_{n-1}D_0 - \frac{g_{0,n-1}}{g_{n-1,0}}D_0D_{n-1} = \frac{x_0}{g_{n-1,0}}D_{n-1} - \frac{x_{n-1}}{g_{n-1,0}}D_0. \tag{81}$$

Then using the involution property of the algebra one can identify

$$D_{n-1} = D_0^+. \tag{82}$$

Hence the pair a_0^+ and a_0 is singled out of the corresponding $n-1$ relations in the first line of equations (63), and the rescaled elements are defined by

$$\begin{aligned} D_0 &= \frac{x_0}{g_{n-1,0}}\left(\frac{1}{1-r_0} + \frac{a_0^+}{\sqrt{1-r_0}}\right) \\ D_{n-1} &= -\frac{x_{n-1}}{g_{n-1,0}}\left(\frac{1}{1-r_0} + \frac{a_0}{\sqrt{1-r_0}}\right) \end{aligned} \tag{83}$$

which satisfies (81) with

$$r_0 = \frac{g_{0,n-1}}{g_{n-1,0}}. \tag{84}$$

The rest of the generators D_k, $k = 1, 2, \ldots, n-2$ are to be identified with the remaining $n-2$ creation operators a_k^+

$$D_k = a_k^+ \tag{85}$$

for k different from the fixed index $i = 0$. The diffusion algebra is obtained from the deformed Heisenberg commutation relations (63)

$$\begin{aligned} &D_{n-1}D_0 - \frac{g_{0,n-1}}{g_{n-1,0}}D_0D_{n-1} = \frac{x_0}{g_{n-1,0}}D_{n-1} - \frac{x_{n-1}}{g_{n-1,0}}D_0 \\ &D_0D_k - q_kD_kD_0 = -\frac{x_0}{g_k}D_k \\ &D_kD_{n-1} - q_kD_{n-1}D_k = \frac{x_{n-1}}{g_k}D_k \\ &D_kD_l - q_{kl}^{-1}D_lD_k = 0 \end{aligned} \tag{86}$$

where $k, l = 1, 2 \ldots n-2$ and

$$q_k = \frac{g_{k0}}{g_{0k}} = \frac{g_{n-1,k}}{g_{k,n-1}} \tag{87}$$

$$g_k = g_{0k} = g_{k,n-1} \qquad g_{0k} - g_{k0} = g_{k,n-1} - g_{n-1,k} = g_{0,n-1} - g_{n-1,0}. \tag{88}$$

The symmetry is reduced with different positive deformation parameters r_0, q_k and q_{kl}. The last relation in (86) with the constraint

$$\frac{g_{kl}}{g_{lk}} = q > 0 \qquad k < l \tag{89}$$

leads to a one-parameter $GL_q(n-2)$ quantization and its dual form as a deformation of the UEA of $gl(n-2)$ with an R-matrix satisfying the Yang–Baxter equation. The matrices D_0 and D_{n-1} and the unit generate the deformed universal enveloping algebra of $gl(2)$. The symmetry in the bulk is thus reduced to the q-direct sum of two quantum algebras $gl_q(n-2) \otimes gl_q(2)$, each being deformed with a different deformation parameter and this is the hidden symmetry algebra in the presence of two x linear terms.

To completely solve the algebraic relations with x-terms present we have to find the boundary vectors for the left and right transition rate matrices. We present now an explicit solution of the boundary eigenvalue problem for the case of $n = 2$ and 3 species. The generalization to higher n is, in principle, straightforward and is only a longer mathematical exsercise.

The case $n = 2$. This is the partially asymmetric exclusion model and its exact solution using a deformed oscillator algebra is known [30, 31]. We comment upon it here for the reason of pointing out the relation to the $sl_q(2)$ R-matrix and its relevance for the integrability, and to show the difference in the solution of the boundary problem compared to $n > 2$. The model is defined by

$$\begin{aligned} &D_1 D_0 - r D_0 D_1 = x_0 D_1 - x_1 D_0 \\ &\langle w|(L_1^0 D_0 - L_0^1 D_1 + x_1) = 0 \\ &(-R_1^0 D_0 + R_0^1 D_1 - x_0)|v\rangle = 0 \end{aligned} \tag{90}$$

with $x_0 + x_1 = 0$ and $r = \frac{g_{01}}{g_{10}}$. Its solution is given by

$$\begin{aligned} D_0 &= \frac{x_0}{g_{10}} \left(\frac{1}{1-r} + \frac{1}{\sqrt{1-r}} a^+ \right) \\ D_1 &= -\frac{x_1}{g_{10}} \left(\frac{1}{1-r} + \frac{1}{\sqrt{1-r}} a \right) \end{aligned} \tag{91}$$

where

$$aa^+ - r a^+ a = 1 \tag{92}$$

and their action on the basis vectors $|n_1\rangle$ labelled by an occupation number $n_1 \equiv m$, $m = 0, 1, 2, \ldots$ is given by the expressions (69)–(71). The explicit representation of the matrices D_0 and D_1 in the basis $|m\rangle$ lead to a corresponding representation of the boundary vectors, namely

$$\langle m|v\rangle = \frac{v_m}{\prod_{i=1}^m \sqrt{1-r^i}} \tag{93}$$

and

$$\langle w|m\rangle = \frac{w_m}{\prod_{i=1}^m \sqrt{1-r^i}}. \tag{94}$$

With the normalization

$$\langle 0|v\rangle = 1 \qquad \langle w|0\rangle = 1 \tag{95}$$

one finds for the model parameters

$$\begin{aligned} v_1 &= 1 + \frac{R_1^0}{R_0^1} + \frac{g_{10} - g_{01}}{R_0^1} \\ w_1 &= 1 + \frac{L_0^1}{L_1^0} + \frac{g_{10} - g_{01}}{L_1^0} \end{aligned} \tag{96}$$

and all the other coefficients v_i, w_i for $i > 1$ being expressed through recurrence relations by v_1 and w_1.

We write now explicitly the corresponding transition matrices

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -g_{01} & g_{10} & 0 \\ 0 & g_{01} & -g_{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{97}$$

and

$$H^L = \begin{pmatrix} -L_1^0 & L_0^1 \\ L_1^0 & -L_0^1 \end{pmatrix} \qquad H^R = \begin{pmatrix} -R_1^0 & R_0^1 \\ R_1^0 & -R_0^1 \end{pmatrix}. \tag{98}$$

Consider now the R-matrix of the two-parameter standard $GL_{p,q}(2)$ deformation [28]

$$\check{R}(p,q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\frac{1}{pq} & \frac{1}{p} & 0 \\ 0 & \frac{1}{q} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{99}$$

that satisfies the quantum Yang–Baxter equation

$$\check{R}_{12}(p,q)\check{R}_{23}(p,q)\check{R}_{12}(p,q) = \check{R}_{23}(p,q)\check{R}_{12}(p,q)\check{R}_{23}(p,q). \tag{100}$$

One can shift the R-matrix by the 4×4 unit matrix to obtain an R-matrix which at the particular point $(p,q) = (1,q)$ has the property of an intensity matrix

$$\check{R}(1,q) - 1_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{101}$$

The matrix $(\check{R}(p,q) - 1_4) \equiv \check{R}'$ satisfies a modified Yang–Baxter equation

$$\begin{aligned} &\check{R}'_{12}(p,q)\check{R}'_{23}(p,q)\check{R}'_{12}(p,q) - \check{R}'_{23}(p,q)\check{R}'_{12}(p,q)\check{R}'_{23}(p,q) \\ &\quad = \check{R}'_{12}(p,q) - \check{R}'_{23}(p,q). \end{aligned} \tag{102}$$

It is readily seen that the bulk transition rate matrix (97) of the asymmetric exclusion model is up to the factor g_{10} equivalent to the $gl_q(2)$ R-matrix $\check{R}(1,q)$ with $q^{-1} = \frac{g_{01}}{g_{10}}$. The symmetry algebra of the model without the x-terms on the right-hand side and with the terms present is $sl_q(2) \otimes u(1)$.

The case $n = 3$. The algebra for the three species problem, generated by the unit and D_0, D_1, D_2, with the choice

$$x_1 = 0 \qquad x_0 + x_2 = 0 \tag{103}$$

has the form

$$\begin{aligned} &g_{01}D_0D_1 - g_{10}D_1D_0 = -x_0D_1 \\ &g_{02}D_0D_2 - g_{20}D_2D_0 = x_2D_0 - x_0D_2 \\ &g_{12}D_1D_2 - g_{21}D_2D_1 = x_2D_1. \end{aligned} \tag{104}$$

The UEA generated by D_0, D_1 and D_2 belongs to the UEA of a deformed Heisenberg algebra of two oscillators a_0^+, a_1^+ which we write explicitly:

$$\begin{aligned} &a_0^+a_1^+ = q_{01}^{-1}a_1^+a_0^+ \\ &a_0a_1 = q_{01}^{-1}a_1a_0 \\ &a_0a_1^+ = q_{01}a_1^+a_0 \\ &a_1a_0^+ = q_{01}^{-1}a_0^+a_1 \\ &a_0a_0^+ = r_0a_0^+a^0 + 1 \\ &a_1a_1^+ = r_1a_1^+a_1 + 1. \end{aligned} \tag{105}$$

Then taking the subset a_0, a_0^+, a_1^+ and perfoming the corresponding shifts (83) we identify

$$
\begin{aligned}
D_0 &= \frac{x_0}{g_{20}}\left(\frac{1}{1-r_0} + \frac{a_0^+}{\sqrt{1-r_0}}\right) \\
D_1 &= a_1^+ \\
D_2 &= -\frac{x_2}{g_{20}}\left(\frac{1}{1-r_0} + \frac{a_0}{\sqrt{1-r_0}}\right) \\
r_0 &= \frac{g_{02}}{g_{20}} \qquad \frac{g_{01}}{g_{10}} = \frac{g_{12}}{g_{21}} = q_{01}.
\end{aligned}
\tag{106}
$$

It is straightforward to verify that this choice of the matrices solves the diffusion algebra (104) provided

$$
g_{01} - g_{10} = g_{02} - g_{20} = g_{12} - g_{21}. \tag{107}
$$

With the algebraic solution at hand there is no essential difficulty in finding the consistent solution of the boundary problem for $n = 3$ and in general for any n. The explicit representation of the matrices D in the two-oscillator basis $|n_1, n_2\rangle$ give a corresponding representation of the boundary vectors

$$
\begin{aligned}
|v\rangle &= \sum_{n_1,n_2} |n_1, n_2\rangle\langle n_1, n_2|v\rangle \\
\langle w| &= \sum_{n_1,n_2} \langle w|n_1, n_2\rangle\langle n_1, n_2|.
\end{aligned}
\tag{108}
$$

The difference to $n = 2$ is that the parameters x_0 and x_2 provide a matching condition for the eigenvalue problem and the boundary vectors being specified in this case by the linear forms (26) and (27) with $x_1 = 0$ will have much more complicated expressions for their coefficients. The diffusion algebra matrices generate the UEA $gl_q(2) \otimes u(1)$ which is the quantum symmetry algebra of the diffusion model.

3.3. Algebras with two x-dependent terms and only $g_{kj} \neq 0$

We rewrite the algebra (86) with two x-terms in a slightly different form

$$
g_{n-1,0}D_{n-1}D_0 - g_{0,n-1}D_0D_{n-1} = x_0D_{n-1} - x_{n-1}D_0 \tag{109}
$$

$$
\begin{aligned}
&g_{0k}D_0D_k - g_{k0}D_kD_0 = -x_0D_k \\
&g_{k,n-1}D_kD_{n-1} - g_{n-1,k}D_{n-1}D_k = x_{n-1}D_k \\
&D_kD_l - q_{kl}^{-1}D_lD_k = 0
\end{aligned}
\tag{110}
$$

where $k, l = 1, 2, 3, \ldots, n-2$. With the choice $g_{0k} = g_{k,n-1} \equiv g_k = 0$ the algebra becomes

$$
D_{n-1}D_0 - \frac{g_{0,n-1}}{g_{n-1,0}}D_0D_{n-1} = \frac{x_0}{g_{n-1,0}}D_{n-1} - \frac{x_{n-1}}{g_{n-1,0}}D_0 \tag{111}
$$

$$
g_{k0}D_kD_0 = x_0D_k \qquad g_{n-1,k}D_{n-1}D_k = -x_{n-1}D_k \tag{112}
$$

$$
D_kD_l - q_{kl}^{-1}D_lD_k = 0. \tag{113}
$$

Assuming that co-ordinates (at least one) D_k of the $(n-2)$-dimensional non-commutative space are invertible matrices we can obtain a particular solution of equation (112)

$$
D_0 = \frac{x_0}{g_{k0}}D_k^{-1}D_k \qquad D_{n-1} = -\frac{x_{n-1}}{g_{n-1,k}}D_kD_k^{-1} \tag{114}
$$

where the rates g_{0k}, respectively $g_{n-1,k}$, are constant values for all $k = 1, 2, \ldots, n-2$. Inserting this solution back into (109) we find a relation for the rates

$$
g_{n-1,0} - g_{0,n-1} = g_{k0} + g_{n-1,k}. \tag{115}
$$

To complete the particular solution we write the realization of a creation operator with (finite- or infinite-dimensional) matrices with non-zero determinant, namely

$$a_k^+ = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & r_k & 0 & \dots & 0 \\ 0 & 0 & r_k^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & r_k^{n-2} \end{pmatrix} \tag{116}$$

where $r_k = q_{kl}^{-1} \equiv q^{-1}$. A more general solution (not necessarily with invertible matrices D_k) is obtained with the realization of the $n-2$ co-ordinates D_k given by equations (61), (69)–(71) and one-dimensional D_0 and D_{n-1}, provided the relation (115) for the rates holds. The elements D_0 and D_{n-1} generate the algebra $u(1) \oplus u(1)$ and the symmetry of the model is $U(1) \otimes U(1) \otimes GL_q(n-2)$.

To proceed with the analogy of a diffusion Fock algebra with the n-dimensional multiparameter noncommutative space we note that using its invariance with respect to a shift of all its co-ordinates we can obtain an algebra that is consistent with braid associativity and contains more than just two c-numbers x. However, a shift of more than one co-ordinate in the quantum space defining commutations relations of only creation operators will produce *quadratic* x-terms in the algebra and applications of such diffusion algebras are not known.

3.4. Algebras with quadratic x-terms

We consider the n-dimensional quantum plane

$$g_{ik} D_i D_k - g_{ki} D_k D_i = 0 \tag{117}$$

with $i < k$ and shift in general all (or at least two of) the co-ordinates D_i, $i = 0, 1, 2, \dots, n-1$ by the unit matrix proportional to the c-numbers x_i. With the assumption $g_{ik} - g_{ki} = \text{const}$ this has the form

$$D_i = D_i' - \frac{x_i}{g_{ik} - g_{ki}}. \tag{118}$$

Hence one obtains the algebra

$$g_{ik} D_i D_k - g_{ki} D_k D_i = x_k D_i - x_i D_k + \frac{x_i x_k}{g_{ik} - g_{ki}} \tag{119}$$

which is consistent with braid associativity and to which the $GL_q(n)$ R-matrix corresponds.

To summarize we have shown that the quadratic Fock algebras discussed within the matrix product ground state approach to stochastic diffusion processes reveal hidden symmetries of these models. The quadratic algebras of n-species diffusion systems either generate the UEA of $gl(n)$, or define an n-dimensional noncommutative space with the $GL_q(n)$ quantum group action as its symmetry and with the dual form as a deformation of the $gl(n)$ UEA. The time-evolution operator governing the stochastic dynamics is thus associated with the $GL_q(n)$ R-matrix satisfying the Yang–Baxter equation and this will allow for a realization of integrable quantum Hamiltonians. When boundary processes take place the $GL_q(n)$ symmetry in the bulk is reduced, the corresponding diffusion algebra being obtained by an appropriate change of basis of the n- (or lower) dimensional quantum space.

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